Norway–Ukrainian cooperation
in
mathematical eduation

The project CPEA-LT-2016/10139 is funded
by Eurasia programme of SIU,
the Norwegian Center for
International Cooperation in Education

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Ukrainian-Norway summer school

Stochastic Analysis, Probability Theory and and Related Topics

Vinnitsa, 2018

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REGULARLY VARYING FUNCTIONS IN PROBABILITY THEORY

Godfrey Harold Hardy (1877–1947)

The man who knew infinity

Tauberian Theorem

If

$$\sum_{n=0}^{\infty} a_n e^{-ny} \sim \frac{1}{y}$$

as $y \downarrow 0$, then

$$\sum_{k=0}^{n} a_k \sim n$$

as $n \longrightarrow \infty$.

Hardy–Littlewood (1915)

$$\sum_{n=2}^{\infty} \Lambda(n) e^{-ny} \sim \frac{1}{y}, \qquad y \downarrow 0,$$

whence

$$\sum_{n \le x} \Lambda(n) \sim x, \qquad x \to \infty.$$

John Edensor Littlewood (1885–1977)

The man who knew infinity

Prime Number Theorem

$$\sum_{n \le x} \Lambda(n) \sim x, \qquad x \to \infty,$$

is equivalent to

$$\pi(x) \sim \frac{x}{\log x}, \qquad x \to \infty.$$

where $\pi(x) = \#(\text{primes} \le x)$.

Riemann Hypothesis

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

$$\zeta(s) = 0 \quad \Longrightarrow \quad \mathrm{Re}(s) = \frac{1}{2}.$$

Jovan Karamata (1902–1967)

Slowly varying functions

A function $L : (0, +\infty) \to (0, +\infty)$ is called *slowly varying* (at infinity) if for all $\lambda > 0$

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1.$$

Example 1. $L(x) = \log(x)$.

Regularly varying functions

A function $R : (0, +\infty) \to (0, +\infty)$ is called *regularly varying* (at infinity) if for all $\lambda > 0$

$$g(\lambda) \stackrel{\text{def}}{=} \lim_{x \to \infty} \frac{R(\lambda x)}{R(x)} \in (0, +\infty).$$

Example 2. $R(x) = x^\rho$, $\rho \in \mathbf{R}$.

Limit function

$$g(\lambda\mu) = \lim_{x\to\infty} \frac{R(\lambda\mu x)}{R(x)}$$

$$= \lim_{x\to\infty} \frac{R(\lambda\mu x)}{R(\mu)} \cdot \frac{R(\mu)}{R(x)}$$

$$= g(\lambda) \cdot g(\mu).$$

Limit function: functional equation

$$\boxed{g(\lambda\mu) = g(\lambda)g(\mu)}$$

Denote

$$\lambda = e^x, \qquad \mu = e^y.$$

Then

$$g(e^{x+y}) = g(e^x)g(e^y).$$

Limit function: functional equation

$$g(e^{x+y}) = g(e^x)g(e^y).$$

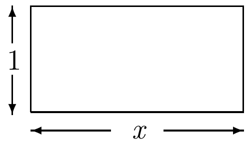
If

$$h(z) \stackrel{\text{def}}{=} \log(g(e^z)),$$

then

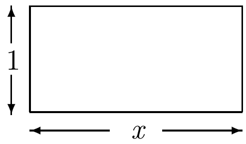
$$\boxed{h(x+y) = h(x) + h(y)}$$

Example 1: square of rectangle



$$h(x) = \text{square}(\text{rectangle } 1 \times x)$$

Example 1: square of rectangle



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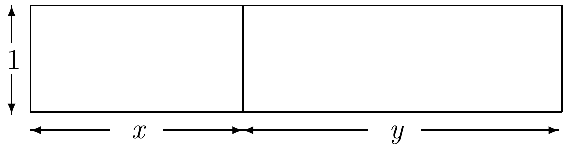
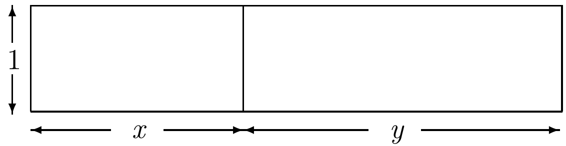


$$h(x+y) = h(x) + h(y)$$



$$h(x+y) = h(x) + h(y)$$

Example 2: logarithm

$$\int_1^x \frac{1}{t}\,dt = \log(x), \qquad x > 0.$$

$$f(u) \stackrel{\text{def}}{=} \int_1^u \frac{1}{t}\,dt, \qquad u > 0,$$

Example 2: logarithm

$$\int_1^x \frac{1}{t}\,dt = \log(x), \qquad x > 0.$$

$$f(u) \stackrel{\text{def}}{=} \int_1^u \frac{1}{t}\,dt, \qquad u > 0,$$

Given $u > 0$ and $v > 0$

$$f(u) + f(v) = \int_1^u \frac{1}{t}\,dt + \int_1^v \frac{1}{t}\,dt$$

$$= \int_1^u \frac{1}{t}\,dt + \int_u^{uv} \frac{1}{z}\,dz$$

after the change $z = tu$.

$$f(u) + f(v) = \int_1^{uv} \frac{1}{t}\,dt = f(uv).$$

If $u = e^x$ and $v = e^y$, then

$$f(e^x) + f(e^y) = f(e^{x+y}).$$

$$f(e^x) + f(e^y) = f(e^{x+y}).$$

Finally with $h(z) = f(e^z)$

$$h(x) + h(y) = h(x + y).$$

Example 3: Poisson process

(1) $N_0 = 0$ and each trajectory N_t is a nondecreasing function;

(2) for some $\lambda > 0$, N_t is Poissonian with parameter λt.

(3) $\{N_t, t \geq 0\}$ is a process with independent stationary increments;

Example 3: Poisson process

$$\mathbf{P}(N_t = 0) = e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}\bigg|_{k=0}$$

$$\mathbf{P}(N_t = 0) = e^{-\lambda t}.$$

Example 3: Poisson process

$$\mathbf{P}(N_t = 0) = e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}\bigg|_{k=0}$$

$$\mathbf{P}(N_t = 0) = e^{-\lambda t}.$$

With $P(t) \stackrel{\text{def}}{=} \mathbf{P}(N_t = 0)$

$$
\begin{aligned}
P(s+t) &= \mathbf{P}(N_{s+t} = 0) \\
&= \mathbf{P}(N_s = 0, N_{s+t} - N_s = 0) \\
&= \mathbf{P}(N_s = 0)\,\mathbf{P}(N_{s+t} - N_s = 0) \\
&= \mathbf{P}(N_s = 0)\,\mathbf{P}(N_t = 0) \\
&= P(s)P(t).
\end{aligned}
$$

$$P(s+t) = P(s)P(t)$$

$$h(x) \stackrel{\text{def}}{=} \log(P(x))$$

$$h(s+t) = h(s) + h(t).$$

Augustin-Louis Cauchy (1789–1857)

Cauchy was the man
who taught rigorous analysis
to all of Europe

Judith Grabiner

Cauchy: Cours d'analyse (1821)

$$f(x + y) = f(x) + f(y)$$

Monotone solutions

Theorem 1. *If f is additive and monotone, then there exists $c \in \mathbf{R}$ such that*

$$f(x) = cx.$$

Square of a rectangle

Corollary 1.

$h(x)$ is the square of $1 \times x$ rectangle

$$h(x) = cx \qquad \textit{for some } c \in \mathbf{R}.$$

Corollary 2.

$$f(u) = \int_1^u \frac{1}{t}\,dt,$$

$$h(z) = f(e^z),$$

$$h(z) = cz \qquad \textit{for some } c \in \mathbf{R}.$$

Corollary 3.

$$P(t) = \mathbf{P}(N_t = 0)$$

$$\log(P(t)) = ct \qquad \textit{for some } c \in \mathbf{R}.$$

William Feller (1906–1970)

The notion of regular variation
introduced by J. Karamata in 1930
finds an ever increasing number of
applications in probability theory.

Feller (1967)

Central Limit Theorem

$$\text{Law}\left(\frac{\zeta_n - a_n}{b_n}\right) \Rightarrow \mathfrak{N}(0,1).$$

$$S_n = X_1 + \cdots + X_n$$

Theorem 2. *Let* $\{X_n\}$ *be independent identically distributed random variables and*

$$F = \text{Law } (X_1) \, .$$

Denote

$$V(x) \stackrel{def}{=} \int_{-x}^{x} u^2 \, dF(u).$$

The central limit theorem holds for $\{S_n\}$ *if and only if*

V *is a slowly varying function.*

Lévy theorem

Theorem 3. *Let* $\{X_n\}$ *be independent identically distributed random variables such that*

$$\text{Law}\left(\frac{S_n - a_n}{b_n}\right) \Rightarrow G$$

for some distribution function G *and some sequences of real numbers* $\{a_n\}$ *and* $\{b_n\}$*. Then* G *is stable.*

Notation

$$F \in \mathfrak{D}(G)$$

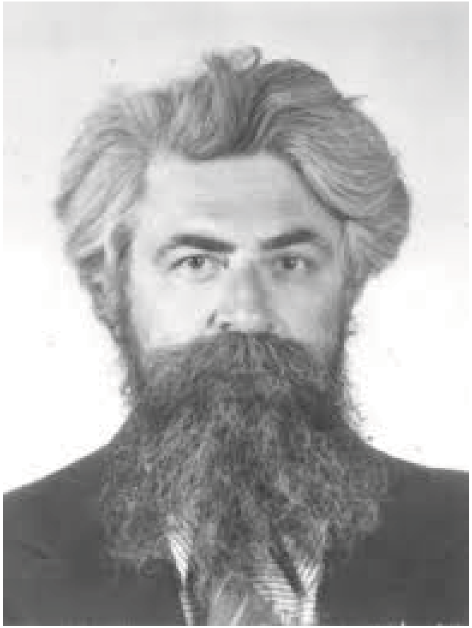
$$T(x) \stackrel{\text{def}}{=} F(-x) + 1 - F(x), \qquad x \geq 0.$$

Lévy–Gnedenko–Kolmogorov theorem

Theorem 4. *Let* $\{X_n\}$ *be independent identically distributed random variables with the distribution function* F *and let* G *be a non-Gaussian stable distribution function. Then* $F \in \mathfrak{D}(G)$ *if and only if*

$$T \text{ is a regularly varying function.}$$

Gleb Sakovich (1932–1989)



Relative stability

$$\text{Law}\left(\frac{S_n}{b_n}\right) \Rightarrow \text{Law}\ (c) = 1\!\!1_{(c,\infty)}(x).$$

Rogozin theorem

Theorem 5. *Let* $\{X_n\}$ *be nonnegative independent identically distributed random variables with the distribution function* F*. Denote*

$$E(x) \stackrel{def}{=} \int_0^x u\,dF(u).$$

Then $\{S_n\}$ *is relative stable if and only if* E *is a slowly varying function.*

Characterization of geometric distribution

Characterization of Gaussian distribution

Radioactive disintegration