Introduction to Functional Equations

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Notion of Functional Equations

Definition

In the broader sense, functional equation is an equation which involves independent variables, known functions, unknown functions and constants; but we exclude operator equations, differential equations, integral equations and other kinds of equations containing infinitesimal operations.

Change of variables

The basic method that usually is used as a part of a solution of a more complicated problem.
If we have an equation of a type

$$f\left(g(x)\right) = h(x),$$

where functions g and h are some known functions. If function g has an inverse then by change of variable $y = g(x)$ we get

$$f(y) = h\left(g^{-1}(y)\right),$$

which gives the solution.

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$$f(t)=1-(t-1)^2=2t-t^2.$$

Symmetry in equation

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and the answer

$$f(x) = x + c, \; c = f(0).$$

Equations Reducing to Algebraic Systems

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Substract first equation from the second one

$$f\left(\frac{x-1}{x}\right)-f\left(\frac{1}{1-x}\right)=\frac{x-1}{x}-x.$$

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$$\begin{cases} f(x) + f\left(\frac{1}{1-x}\right) = x \\ f\left(\frac{1}{1-x}\right) - f(x) = \frac{x^2-x+1}{x(1-x)} \end{cases}$$

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and for all z, such that $z^2 - z + 1 \neq 0 \ \Leftrightarrow \ z \neq \frac{1 \pm i\sqrt{3}}{2} = z_{1,2}$

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$$\begin{cases} f(z_2) = c; \\ f(z_1) = 1 + z_1 - z_1 f(z_2) = 1 + \frac{1 - i\sqrt{3}}{2}(1 - c). \end{cases}$$

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Therefore, the solution is

$$f(x) = \begin{cases} 1, & z \neq \dfrac{1 \pm i\sqrt{3}}{2}; \\ c, & z = \dfrac{1 + i\sqrt{3}}{2}; \\ 1 + \dfrac{1 - i\sqrt{3}}{2}(1 - c), & z = \dfrac{1 - i\sqrt{3}}{2}. \end{cases}$$

Let g be a known function such that the set $G = \left\{e, g, g^2, \ldots, g^{n-1}\right\}$, where $e(x) = x$ is neutral element, $g^2 = g \circ g$ is a function decomposition of g with itself, is a cyclic group of order n with operation $\circ$. If functional equation

$$a_0(x)f(x) + a_1(x)f(g(x)) + \ldots + a_{n-1}(x)f\left(g^{n-1}(x)\right) = b(x),$$

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Therefore we can get next system

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which solution is

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c) $x=\dfrac{\pi}{2},\ y=t+\dfrac{\pi}{2}$, then $f(t+\pi)+f(-t)=-2f\left(\dfrac{\pi}{2}\right)\sin t$.

Then we obtain linear with respect to $f(t)$, $f(-t)$, $f(t+\pi)$ system

$$\begin{cases} f(t)+f(-t)=2a\cos t; \\ f(t+\pi)+f(t)=0; \\ f(t+\pi)+f(-t)=-2b\sin t, \end{cases}$$

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where $a=f(0)$, $b=f(\pi/2)$. It is easy to solve it and obtain solution

$$f(t)=a\cos t+b\sin t.$$

1st Cauchy Equations

Example 8 (A.L. Cauchy, 1821)

Find all continuous functions f that satisfy the relation

$$f(x+y) = f(x) + f(y),\ x, y \in \mathbb{R}. \tag{8}$$

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Using the last equation we can obtain for any $m \in \mathbb{N}$

$$f(1) = f\left(m\frac{1}{m}\right) = mf\left(\frac{1}{m}\right) \Leftrightarrow f\left(\frac{1}{m}\right) = \frac{1}{m}f(1) = k \cdot \frac{1}{m}$$

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and, therefore, for any $q \in \mathbb{Q}$,

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and, therefore, for any $q \in \mathbb{Q}$,

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3) Denote by $k = f(1)$. Then, if we set $x = y = 1$,

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2nd Cauchy Equations

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Find all continuous functions f that satisfy the relation

$$f(x+y) = f(x)f(y), \ x,y \in \mathbb{R}. \tag{9}$$

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Last equality means that function $\ln f(x)$ satisfies 1st Cauchy equation and we get $f(x) = e^{kx} = a^x$.

3rd and 4th Cauchy Equations

Equation

$$f(xy) = f(x) + f(y),\ x, y \geq 0.$$

is called third (logarithmic) Cauchy equation.

Equation

$$f(xy) = f(x)f(y),\ x, y \geq 0.$$

is called forth (power) Cauchy equation.

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Find all continuous functions f that satisfy the relation

$$f(x+y) = f(x) + f(y) + f(x)f(y),\ x, y \in \mathbb{R}. \tag{10}$$

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or if we define $g(x) = f(x) + 1$

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$$f(x+y)+1=(f(x)+1)(f(y)+1)$$

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which has the solutions $g(x)=0$ or $g(x)=a^x$, $a>0$.

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which has the solutions $g(x) = 0$ or $g(x) = a^x$, $a > 0$. Therefore

$$f(x) = -1 \text{ or } f(x) = a^x - 1.$$

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Find all continuous positive functions f that satisfy the relation

$$f(x^2+y^2) = f(x^2-y^2) + f(2xy), \ x, y \in \mathbb{R}. \qquad (11)$$

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If $x = y = 0$ we obtain $f(0) = 0$. For $x = 0$

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Define

$$s=x^2-y^2,\ t=2xy,\ x>y.$$

Then $s^2+t^2=(x^2+y^2)^2$ and functional equation (11) can be rewritten in the form

$$f\left(\sqrt{s^2+t^2}\right)=f\left(\sqrt{s^2}\right)+f\left(\sqrt{t^2}\right).$$

Finally, let $g(t) = f\left(\sqrt{t}\right)$ and $u = s^2$, $v = t^2$.

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$$g(u+v) = g(u) + g(v),$$

which solution is $g(u) = ku$ and solution of initial functional equation is

$$f(x) = kx^2.$$

Simple interest

Example 12

Let $f(x,t)$ be the interest we receive from the bank after a deposit of an amount x during a period of duration t. Then, if the assumptions of simple interest hold, the function $f(x,t)$ must satisfy the following conditions:

1. At the end of the time period t, we receive the same interest in the following two cases:
   - We deposit the amount $x+y$ in one account;
   - We deposit the amount x in one account, and the amount y in another account.
2. At the end of the time period $t+s$, we receive the same interest in the next two cases:
   - We deposit the amount x during the period of duration $t+s$;
   - We deposit the amount x first during a period of duration t and later for a period of duration s.

From condition 1) it follows that

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$$f(x,s+t) = f(x,s) + f(x,t) \ \Leftrightarrow \ g(s+t) = g(s) + g(t)$$

and the function of simple interest takes the form

$$f(x,t) = k \cdot x \cdot t.$$

Linear Cauchy Equation with weaker conditions

In all the following examples the unique solution is $f(x) = kx$, $x \in \mathbb{R}$.

Example 14 (G.Darboux, 1875)

Find all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at some point x_0 and satisfy Cauchy equation $f(x+y) = f(x) + f(y),\ x, y \in \mathbb{R}$.

It is easy to show that function f will be continuous everywhere under the assumption. Indeed

$$\lim_{t\to x} f(t) = \lim_{t\to x} f(t - x + x_0 + (x - x_0)) =$$

$$= \lim_{t\to x} f(t - x + x_0) + f(x - x_0) = f(x_0) + f(x - x_0) = f(x).$$

Example 15 (G.Darboux, 1880)

Find all functions $f:\mathbb{R}\to\mathbb{R}$ that are monotonically increasing and satisfy Cauchy equation $f(x+y)=f(x)+f(y),\ x,y\in\mathbb{R}$.

Let $x\in\mathbb{R}$ be fixed. Then there exists two rational sequences $\{r_n\}$ and $\{R_n\}$ such that for all n

$$r_n < x < R_n$$

and $\lim\limits_{n\to\infty} r_n = \lim\limits_{n\to\infty} R_n = x$. Then

$$kr_n = f(r_n) \le f(x) \le f(R_n) = kR_n,$$

and obviously $f(x)=kx$.

Example 16 (G.Darboux, 1880)

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that f is nonnegative for sufficiently small x ($f(x) \geq 0$, $x \in (0, \varepsilon)$ for some fixed small $\varepsilon > 0$) and satisfy Cauchy equation $f(x+y) = f(x) + f(y),\ x, y \in \mathbb{R}$.

Under the assumptions in the example we can get that if $f(y) > 0$ for some small y then for any $x > 0$

$$f(x+y) = f(x) + f(y) \geq f(x),$$

and the function f is monotonically increasing. By previous example $f(x) = kx$.

Example 17 (S.Pincherele and U.Amaldi, 1901; E. Picard, 1928)

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that f is bounded on an arbitrary small interval (a,b) and satisfy Cauchy equation
$f(x+y) = f(x) + f(y),\ x,y \in \mathbb{R}$.

Consider function $g(x) = f(x) - xf(1)$. Note that this function also satisties Cauchy equation and it is bounded on (a,b). On the other hand, for any $r \in \mathbb{Q}$

$$g(r) = f(r) - rf(1) = 0,$$

and hence

$$g(x+r) = g(x).$$

Suppose that for some x_0 $g(x_0) = A \neq 0$, then

$$g(nx_0) = ng(x_0) = nA,$$

that contradicts the boundness of g. So $g(x) = 0$ and
$f(x) = xf(1) - kx$.

Linear Cauchy Equation in non-continuous case

We make use of so-called Hamel basis (G. Hamel, 1905). He has proved that there exists a nondenumerable set B of real numbers such that any $x \in \mathbb{R}$ can be represented uniquely as a finite linear composition

$$x = r_1 b_1 + \ldots + r_n b_n, \ b_k \in B \tag{12}$$

with $r_k \in \mathbb{Q}$, $k = \overline{1,n}$.

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Theorem 1

The most general solution which satisfies Cauchy equation $f(x+y) = f(x) + f(y), \;\; x, y \in \mathbb{R}$, can be constructed if we choose arbitrary values of f for the points of a Hamel basis B and define value of f for any $x \in \mathbb{R}$ that has representation (12) by

$$f(x) = r_1 f(b_1) + \ldots + r_n f(b_n).$$

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THANK YOU FOR YOUR ATTENTION!